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Lipschitz and Strong Unicity Constants for Changing Dimension

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1. INTRODUCTION

Let $C(I)$ denote the set of continuous, real-valued functions on the interval $I = [-1, 1]$, and let $\mathscr{P}_{n+1} \subseteq C(I)$ be a Haar subspace of dimension $n + 1$. Let $\|\cdot\|$ denote the uniform norm on $C(I)$. For $f \in C(I)$ with best uniform approximation $T_n(f)$ from $\mathscr{P}_{n+1}$ there are positive constants $\gamma_n(f)$ and $\lambda_n(f)$ such that for any $p \in \mathscr{P}_{n+1}$ and any $g \in C(I)$,

$$\|f - p\| \geqslant \|f - T_n(f)\| + \gamma_n(f) \|p - T_n(f)\|, \tag{1.1}$$

and

$$\|T_n(f) - T_n(g)\| \leqslant \lambda_n(f) \|f - g\|. \tag{1.2}$$

Inequality (1.1) is the well-known strong unicity Theorem [3, p. 80], and inequality (1.2) is the Freud theorem [3, p. 82]. A number of recent papers [1, 2, 4–6, 8, 10] have examined the constants $\gamma_n(f)$ and $\lambda_n(f)$. In particular, for fixed n, Bartelt [1] and Cline [4] show that $\gamma = \gamma(f)$ may actually be chosen independent of f if the interval I is replaced by a finite point set X. Henry and Schmidt [6] show for compact subsets $\Gamma \subseteq C(I)$ with $\Gamma \cap \mathscr{P}_{n+1} = \varnothing$ that the constant $\lambda(f)$ in (1.2) may be replaced by a constant λ_Γ and that (1.2) remains valid for all $f \in \Gamma$ and $g \in C(I)$. Bartelt [1] gives conditions that ensure for sequences $\{f_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} \|f - f_j\| = 0$ that $\lim_{j\to\infty} \lambda(f_j) = \lambda(f)$.

For fixed f and n, Henry and Roulier [5] investigate the behavior of $\lambda(f)$ for changing intervals.

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For fixed f and changing n, Poreda [10] investigates the properties of the sequence $\{\gamma_n(f)\}_{n=0}^{\infty}$. The purpose of the present paper is to extend the investigations initiated in [10]. Thus we are interested in the behavior of the sequences $\{\gamma_n(f)\}_{n=0}^{\infty}$ and $\{\lambda_n(f)\}_{n=0}^{\infty}$ for appropriate functions $f \in C(I)$.

2. Strong Unicity Constants

Let $f \in C(I)$, let $\gamma_n(f)$ be the largest constant for which (1.1) is valid for all $p \in \mathscr{P}_{n+1}$, and let $\lambda_n(f)$ be the smallest constant for which (1.2) is true for all $g \in C(I)$. Define $S(\mathscr{P}_{n+1}) = \{p \in \mathscr{P}_{n+1}: \|p\| = 1\}$. Then it is known ([1, 2], and in particular [8, Lemma 1]) that

$$\gamma_n(f) = \inf_{p \in S(\mathscr{P}_{n+1})} \max_{x \in E_{n+1}(f)} \operatorname{sgn}[f(x) - T_n(f)(x)]\, p(x), \tag{2.1}$$

where

$$E_{n+1}(f) = \{x \in I: |f(x) - T_n(f)(x)| = \|f - T_n(f)\|\}.$$

It is also known that

$$\lambda_n(f) \leqslant 2[\gamma_n(f)]^{-1}, \qquad n = 0, 1, 2, \ldots, \tag{2.2}$$

and that

$$0 < \gamma_n(f) \leqslant 1, \qquad n = 0, 1, 2, \ldots; \tag{2.3}$$

see [3, pp. 80, 82].

Let

$$M_n(f) = [\gamma_n(f)]^{-1}. \tag{2.4}$$

In [10] Poreda poses the following problem: For what functions $f \in C(I)$ is the sequence

$$\{M_n(f)\}_{n=0}^{\infty} \tag{2.5}$$

bounded? We note from (2.2) and (2.4) that if (2.5) is bounded, then

$$\{\lambda_n(f)\}_{n=0}^{\infty} \tag{2.6}$$

will also be a bounded sequence.

It is clear if $\mathscr{P}_{n+1} = \pi_n$, the set of algebraic polynomials of degree at most n, and if f is any polynomial, then (2.5) is a bounded sequence. The next theorem, due to Poreda [10], shows that there exist functions $f \in C(I)$ for which (2.5) is an unbounded sequence. Hereafter we will assume the approximating class is π_n.

THEOREM 1. *There exists a function $f \in C(I)$ such that* (2.5) *is unbounded.*

Poreda actually established by a clever construction that there exists a subsequence $\{M_{n_i}(f)\}_{i=1}^{\infty}$ of (2.5) such that

$$\lim_{i \to \infty} M_{n_i}(f) = +\infty. \tag{2.7}$$

Poreda claims, however, to have established that

$$\lim_{n \to \infty} M_n(f) = +\infty, \tag{2.8}$$

but it appears to the present authors that to establish (2.8), Poreda makes use of a remark that appears in [10], namely that

$$M_{n+1}(f) \geqslant M_n(f), \qquad n = 0, 1, \dots, \tag{2.9}$$

for any $f \in C(I)$. However, inequality (2.9) is, in general, false, as the following example demonstrates.

EXAMPLE 1. Let $f(x) = x^3$, $I = [-1, 1]$, and suppose that approximation is from π_2 and π_3, respectively. It is easy to see that $T_2(f)(x) = 3x/4$ and that $T_3(f)(x) = x^3$. Then

$$\gamma_2(f) = \inf_{\substack{p \in \pi_2 \\ p \neq T_2(f)}} \frac{\| f - p \| - \| f - T_2(f) \|}{\| p - T_2(f) \|}. \tag{2.10}$$

But if $p(x) = x$, the quotient in (2.10) is less than one, and hence $\gamma_2(f) < 1$. Thus by (2.4) $M_2(f) > 1$. However, since $f \in \pi_3$, $M_3(f) = 1$.

In the remainder of this section we prove the existence of functions $f \in C(I)$ for which (2.8) is true by utilizing techniques that are entirely different than those employed by Poreda. These techniques will yield a class of functions with properties quite different from those possessed by the function constructed in [10]. The following two theorems are needed in the subsequent analysis.

THEOREM 2. *Suppose that approximation is from π_n, that $f \in C[-1, 1]$, and that $f'' \in C(-1, 1)$. Further assume that $f^{(n+1)}$ is positive and strictly increasing on $(-1, 1)$. Let $-1 \leqslant x_{0n} < x_{1n} < \cdots < x_{nn} < x_{n+1,n} \leqslant 1$ be the ordering of $E_{n+1}(f)$. Then $z_{k,n} < x_{k,n} < z_{k+1,n}$, where*

$$z_{kn} = \cos\left(\frac{n+1-k}{n+1}\right)\pi, \qquad k = 1, 2, \dots, n. \tag{2.11}$$

Theorem 2 is actually a special case of Theorem 3.3 in [11]. See also [9, p. 101].

THEOREM 3 (Cline [4]). *Let $f \in C[-1, 1]$ with $f \notin \pi_n$. Let $T_n(f) \in \pi_n$ be the best approximation to f, and for any Chebyshev alternation $\{x_{kn}\}_{k=0}^{n+1}$ for $f - T_n(f)$ define $q_{in} \in \pi_n$ by $q_{in}(x_{kn}) = \operatorname{sgn}[f(x_{kn}) - T_n(f)(x_{kn})]$, $k = 0, 1,..., n + 1$, $k \neq i$, and $i = 0,..., n + 1$. Then*

$$M_n(f) \leq \max_{0 \leq i \leq n+1} \|q_{in}\|. \tag{2.12}$$

Remark. If, as in Theorem 2, $E_{n+1}(f)$ contains exactly $n + 2$ points, then $E_{n+1}(f)$ is a Chebyshev alternation. In this case, it is easy to show that

$$M_n(f) = \max_{0 \leq i \leq n+1} \{\|q_{in}\|\}. \tag{2.13}$$

To see this, one uses Theorem 5 and Lemma 3 of [4], tohether with the observation that $M_n(f) = \bar{K}$ as given in expression (4) of [4] in this case. This latter observation is a direct consequence of (2.1) and (2.4).

We utilize Eq. (2.13) to establish the main theorem of this section.

THEOREM 4. *Suppose there exists an integer $N > 0$ and a real number $\alpha > 0$ such that f satisfies the hypotheses of Theorem 2 for all $n \geq N$, and, such that*

$$f^{(n+1)}(\xi)/f^{(n+1)}(\eta) \geq \alpha > 0,$$

for all $\xi, \eta \in [-1, 1]$ and for all $n \geq N$. Then

$$\lim_{n \to \infty} M_n(f) = +\infty.$$

Proof. By the remark following Theorem 3 we note that

$$M_n(f) \geq \|q_{0n}\|. \tag{2.14}$$

But

$$q_{0n}(x_{kn}) = \operatorname{sgn}[f(x_{kn}) - T_n(f)(x_{kn})],$$

$k = 1, 2,..., n + 1$. Thus $q_{0n}(x)$ interpolates the function

$$g(x) = \frac{f(x) - T_n(f)(x)}{\|f - T_n(f)\|} \tag{2.15}$$

at $\{x_{kn}\}_{k=1}^{n+1}$. Let

$$e_n(f) = \|f - T_n(f)\|. \tag{2.16}$$

Then the classical remainder theorem of interpolation theory [3, p. 60] implies that

$$g(x) - q_{0n}(x) = \frac{g^{(n+1)}(\xi_x)(x - x_{1n})(x - x_{2n}) \cdots (x - x_{nn})(x - x_{n+1,n})}{(n + 1)!}, \tag{2.17}$$

where $-1 \leqslant \xi_x \leqslant 1$. Then (2.15), (2.16), and (2.17) imply that

$$| g(x)| + | q_{0n}(x)| \geqslant \frac{| f^{(n+1)}(\xi_x) | \, | x - x_{1n} | \cdots | x - x_{nn} | \, | x - x_{n+1,n} |}{e_n(f)(n+1)!} \tag{2.18}$$

But from [9, p. 78],

$$e_n(f) = \frac{| f^{(n+1)}(\eta) |}{2^n(n+1)!},$$

where $-1 \leqslant \eta \leqslant 1$. Therefore, (2.15) and (2.18) imply that

$$1 + \| q_{0n} \| \geqslant \left| \frac{f^{(n+1)}(\xi_x)}{f^{(n+1)}(\eta)} \right| 2^n \, | x - x_{1n} | \cdots | x - x_{nn} | \, | x - x_{n+1,n} | \tag{2.19}$$

for each $x \in [-1, 1]$.

Inequality (2.19) and the hypothesis of Theorem 4 now imply that

$$1 + \| q_{0n} \| \geqslant \alpha 2^n \, |(-1 - x_{1n})(-1 - x_{2n}) \cdots (-1 - x_{nn})(-1 - x_{n+1,n})|$$

for $n \geqslant N$. An application of Theorem 2 yields

$$\begin{aligned} 1 + \| q_{0n} \| &\geqslant \alpha \, | 2^n(-1 - z_{1n})(-1 - z_{2n}) \cdots (-1 - z_{nn})| \\ &= \alpha \, | C'_{n+1}(-1)|/(n+1), \end{aligned} \tag{2.20}$$

where C_{n+1} is the Chebyshev polynomial of degree $n + 1$. But it is well known that $| C'_{n+1}(1)| = (n+1)^2$. Therefore (2.20) implies for $n \geqslant N$ that

$$1 + \| q_{0n} \| \geqslant \alpha(n+1). \tag{2.21}$$

Finally (2.21) and (2.13) imply that

$$1 + M_n(f) \geqslant \alpha(n+1), \qquad n \geqslant N,$$

and consequently

$$\lim_{n\to\infty} M_n(f) = +\infty. \quad \blacksquare$$

EXAMPLE 2. Let $f_1(x) = e^x$, $f_2(x) = (x+1)\, e^{x+1}$, $I = [-1, 1]$. Then Theorem 4 implies that $\lim_{n\to\infty} M_n(f_i) = +\infty$, $i = 1, 2$.

The results of Theorem 4 are perhaps surprising when compared with the results of Poreda. In particular, the construction in [10] requires for the indices $\{n_i\}_{i=1}^{\infty}$ that $E_{n_i+1}(f) \subset (a, b) \subset [-1, 1]$, where containment is proper, and the interval (a, b) does not depend on i. In contrast, functions satisfying Theorem 4 have Chebyshev alternation sets $E_{n+1}(\pi)$ that behave similarly to the extreme points of the $(n+1)$st-degree Chebyshev polynomial C_{n+1}. Thus it seems plausible (at least to these authors) that (2.5) may be bounded only for polynomial functions.

We conclude this section by considering a second theorem due to Poreda [10] for the case $n = 1$.

THEOREM 5. *Let B denote the unit ball of $C(I)$. Then for $n \geqslant 1$, the set $\{M_n(f)\}_{f \in B}$ is not bounded.*

In light of Example 1, Poreda establishes this theorem only for $n = 1$ although he states it for $n \geqslant 1$. However, Cline [4, Theorem 4] establishes that for fixed $n \geqslant 2$ and for any $\epsilon > 0$, there exist functions g_ϵ and f_ϵ, $\|f_\epsilon\| = 1$, continuous on I, with corresponding best approximations $T(g_\epsilon)$ and $T(f_\epsilon)$, satisfying

$$\frac{\| T(g_\epsilon) - T(f_\epsilon)\|}{\| g_\epsilon - f_\epsilon \|} \geqslant \frac{1}{\epsilon}.$$

It follows immediately that

$$\sup_{f \in B} \{\lambda_n(f)\} = +\infty, \qquad n = 2, 3, \ldots .$$

Theorem 5 now follows for $n = 2, 3, \ldots$, from (2.2) and (2.4).

3. Lipschitz Constants

Theorem 4 of Section 2 says that (2.5) may be unbounded for functions that are restrictions of entire functions to the segment $[-1, 1]$ of the complex plane. On the basis of inequality (2.2) it remains to establish a companion result for sequence (2.6). The techniques employed in the previous section do not appear applicable in the Lipschitz constant setting, since $2M_n(f)$ is merely an upper bound for $\lambda_n(f)$, $n = 0, 1, \ldots$. In particular

$$\lambda_n(f) = \sup_{\substack{g \in C(I) \\ g \neq f}} \frac{\| T_n(f) - T_n(g)\|}{\| f - g \|}. \tag{3.1}$$

No alternate representation of $\lambda_n(f)$ (like (2.1) for the strong unicity constant) is known to the authors of this paper.

Thus we construct an $f \in C(I)$ and a sequence $\{ g_{n_i}\}_{i=1}^{\infty} \subseteq C(I)$ such that

$$\lim_{i \to \infty} \frac{\| T_{n_i}(f) - T_{n_i}(g_{n_i})\|}{\| f - g_{n_i} \|} = +\infty. \tag{3.2}$$

The construction of the function f is based on the construction in [10], and f will be the restriction of an entire function to the segment $[-1, 1]$ of the complex plane.

Let $[x_1, x_\infty]$ be properly contained in $[0, \frac{1}{2}]$, and let $\{x_k\}_{k=1}^{\infty} \subset (0, \frac{1}{2})$ be a monotone sequence converging to x_∞. Let Q_{n_i} be a polynomial satisfying

$$
\begin{aligned}
&\text{(i)} \quad Q_{n_i}(-1) = -1/2^{i+1},\\
&\text{(ii)} \quad Q_{n_i}(-\tfrac{1}{2}) = 0,\\
&\text{(iii)} \quad Q_{n_i}(0) = 1/2^{i+1},\\
&\text{(iv)} \quad Q_{n_i}(x_k) = (-1)^k/2^i \qquad \text{for} \quad k = 1, 2,\ldots, 2n_i + 2,\\
&\text{(v)} \quad Q_{n_i}(x_{2n_i+3}) = (-1)^{2n_i+3}/2^{i+1},\\
&\text{(vi)} \quad Q_{n_i}(\tfrac{1}{2}) = 0,\\
&\text{(vii)} \quad Q_{n_i}(1) = 1/2^{i+1},
\end{aligned}
\tag{3.3}
$$

(viii) $Q_{n_i}(x)$ is monotone on the intervals $[-1, -\frac{1}{2}]$, $[-\frac{1}{2}, 0]$, $[0, x_1]$, $[x_{k-1}, x_k]$, $k = 2,\ldots, 2n_i + 3$, $[x_{2n_i+3}, \frac{1}{2}]$, and $[\frac{1}{2}, 1]$.

We note a theorem of Wolibner [12] (see also [10]) assures the existence of a polynomial Q_{n_i} satisfying (3.3). To define n_i set $n_1 = 2$ and let n_{i+1} be the degree of Q_{n_i}, for $i = 1, 2,\ldots$. Now define $Q_{n_i}^*$ by

$$Q_{n_i}^*(x) = Q_{n_i}(x)/(n_{i+1}\,!)^2.$$

Let

$$f(x) = \sum_{i=1}^{\infty} Q_{n_i}^*(x)$$

and

$$p_{n_i}(x) = \sum_{j=1}^{i-1} Q_{n_j}^*(x). \tag{3.5}$$

Then as in [10], it is easily shown that $p_{n_i}(x)$ is the best approximation from π_{n_i} to f on $[-1, 1]$. Now consider the complex functions

$$f(z) = \sum_{i=1}^{\infty} Q_{n_i}^*(z) \tag{3.6}$$

and

$$u_i(z) = p_{n_i}(z) = \sum_{j=1}^{i-1} Q_{n_j}^*(z). \tag{3.7}$$

Let $\mathscr{E}_\rho$ denote interior and boundary of the ellipse with foci at ± 1 and with semi-axes $a = \frac{1}{2}(\rho + \rho^{-1})$, $b = \frac{1}{2}(\rho - \rho^{-1})$. Then a theorem of Bernstein [7, p. 42] implies for any polynomial p_n,

$$| p_n(z)| \leqslant M\rho^n, \qquad z \in \mathscr{E}_\rho, \qquad \rho > 1, \tag{3.8}$$

where $M = \max_{-1\leqslant x\leqslant 1} |p_n(x)|$. Consequently (3.8) and the definition of Q_{n_i} imply

$$|f(z) - u_i(z)| \leqslant \sum_{j=i}^{\infty} |Q^*_{n_j}(z)|$$

$$= \sum_{j=i}^{\infty} \frac{|Q_{n_j}(z)|}{(n_{j+1}!)^2}$$

$$\leqslant \sum_{j=i}^{\infty} \frac{\max_{-1\leqslant x\leqslant 1} |Q_{n_j}(x)|}{(n_{j+1}!)^2} \rho^{n_{j+1}}$$

$$\leqslant \sum_{j=i}^{\infty} \frac{\rho^{n_{j+1}}}{2^j (n_{j+1}!)^2}. \tag{3.9}$$

Thus for any fixed $\rho > 1$,

$$\lim_{i\to\infty} |f(z) - u_i(z)| = 0$$

uniformly on $\mathscr{E}_\rho$. Since $f(z)$ is the uniform limit of a sequence of analytic functions on $\mathscr{E}_\rho$, $f(z)$ is analytic on $\mathscr{E}_\rho$. This is true for any $\rho > 1$, and consequently $f(z)$ is entire. Thus (3.4) is the restriction of an entire function to the segment $[-1, 1]$ of the complex plane.

We now consider again the quotient in (3.2). Let

$$f_{n_i}(x) = f(x) - p_{n_i}(x) = \sum_{j=i}^{\infty} Q^*_{n_j}(x), \qquad i = 1, 2,\ldots . \tag{3.10}$$

With this notation finding a sequence $\{g_{n_i}\}_{i=1}^{\infty} \subseteq C(I)$ that satisfies (3.2) for the f defined in (3.4) is equivalent to finding a sequence $\{h_{n_i}\}_{i=1}^{\infty} \subseteq C(I)$ satisfying

$$\lim_{i\to\infty} \| T_{n_i}(h_{n_i})\|/\| f_{n_i} - h_{n_i} \| = +\infty. \tag{3.11}$$

This follows from the fact that if $p \in \pi_n$ and if $g \in C(I)$, then $T_n(p + g) = p + T_n(g)$.

We note that the $Q^*_{n_j}$ are monotone in the same sense on $[-1, 0]$, $[0, x_1]$,..., $[x_{2n_i+2}, x_{2n_i+3}]$, $j \geqslant i$, and on $[\frac{1}{2}, 1]$. Also $Q^*_{n_j}(-\frac{1}{2}) = Q^*_{n_j}(\frac{1}{2}) = 0$ for $j \geqslant 1$. Thus f_{n_i} is monotone on these intervals, $i = 1, 2,\ldots$, and $f_{n_i}(-\frac{1}{2}) = f_{n_i}(\frac{1}{2}) = 0$. Furthermore, if

$$\epsilon_i = \sum_{j=i}^{\infty} \frac{1}{2^{j-i}} \frac{1}{(n_{j+1}!)^2}, \tag{3.12}$$

$i = 1, 2, \ldots$, then

$$\begin{aligned} f_{n_i}(-1) &= -(1/2^{i+1})\,\epsilon_i\,, \\ f_{n_i}(0) &= (1/2^{i+1})\,\epsilon_i\,, \\ f_{n_i}(x_k) &= ((-1)^k/2^i)\,\epsilon_i\,, \qquad k = 1, 2, \ldots, 2n_i + 2, \\ f_{n_i}(x_{2n_i+3}) &= -(1/2^{i+1})[2\epsilon_i - (1/(n_{i+1}!)^2)], \\ f_{n_i}(1) &= (1/2^{i+1})\,\epsilon_i\,, \end{aligned} \tag{3.13}$$

and

$$\|f_{n_i}\| = (1/2^i)\,\epsilon_i\,.$$

We now define h_{n_i}, $i = 1, 2, \ldots$, as follows:

$$\begin{aligned} h_{n_i}(x) &= f_{n_i}(x) + c_{n_i}(-\tfrac{1}{2})^{n_i}, \qquad & -1 \leqslant x \leqslant -\tfrac{1}{2}, \\ h_{n_i}(x) &= f_{n_i}(x) + c_{n_i}x^{n_i}, \qquad & -\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2}, \\ h_{n_i}(x) &= f_{n_i}(x) + c_{n_i}(\tfrac{1}{2})^{n_i}, \qquad & \tfrac{1}{2} \leqslant x \leqslant 1. \end{aligned} \tag{3.14}$$

It is clear for any choice of the constant c_{n_i} that h_{n_i} is continuous on $[-1, 1]$.
Equations (3.13) and (3.14) imply that

$$h_{n_i}(x_k) - c_{n_i}x_k^{n_i} = f_{n_i}(x_k) = ((-1)^k/2^i)\,\epsilon_i\,, \qquad k = 1, 2, \ldots, 2n_i + 2. \tag{3.15}$$

We now select c_{n_i} to ensure that

$$\max_{-1 \leqslant x \leqslant 1} |\, h_{n_i}(x) - c_{n_i}x^{n_i} \,| \leqslant \epsilon_i/2^i. \tag{3.16}$$

That such a choice is possible follows from (3.13) and the definition of h_{n_i}. (In fact, $0 < c_{n_i} \leqslant (1/2^{i+1})\,\epsilon_i$ will suffice.) Consequently (3.15), (3.16), and the alternation theorem [3, p. 75] imply that

$$T_{n_i}(h_{n_i})(x) = c_{n_i}x^{n_i}. \tag{3.17}$$

Returning to the quotient in (3.11), we have that

$$\frac{\| T_{n_i}(h_{n_i})\|}{\| f_{n_i} - h_{n_i} \|} = \frac{c_{n_i}}{c_{n_i}(\frac{1}{2})^{n_i}} = 2^{n_i}.$$

This equation implies (3.11). Finally, (3.2) is established for the f of (3.4) and for

$$g_{n_i}(x) = h_{n_i}(x) + p_{n_i}(x), \qquad i = 1, 2, \ldots\,.$$

The above analysis establishes the following theorem.

THEOREM 6. *There exists a function* $f \in C(I)$ *such that* $\sup_n\{\lambda_n(f)\} = +\infty$. *Furthermore, f may be chosen to be the restriction of an entire function to the segment* $[-1, 1]$ *of the complex plane.*

We note since $\lambda_n(f) \leq 2M_n(f)$, $n = 0, 1,...$, that Theorem 6 also implies (2.7).

4. Remarks and Conclusions

Although the question as to whether or not there exists a nonpolynomial function f for which the sequences (2.5) and (2.6) are bounded remains an open question, the results of this paper lead the authors to conjecture that these sequences are bounded only for polynomials.

It is also of interest to determine whether or not these sequences are indeed monotone for nonpolynomial functions.

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